Propagation of Waves in Cylindrical Hard-Walled Ducts with Generally Weak Undulations

Ali H. Nayfeh* and Osama A. Kandil†
Virginia Polytechnic Institute and State University, Blacksburg, Va.

The method of multiple scales is utilized to analyze the wave propagation in cylindrical hard-walled ducts having weak undulations which need not be periodic. Results are presented for two and three interacting modes. In the case of modes traveling in the same direction in a uniform duct, two interacting, spinning or nonspinning modes propagate unattenuated in an undulated duct. Moreover, neither of them can exist without strongly exciting the other. On the other hand, in the case of modes propagating in opposite directions, they may be cut off as a result of the interaction.

I. Introduction

FOR two-dimensional ducts, straightforward expansions of the form $\phi_0+\epsilon\phi_1$ were obtained by Isakovitch[1] for the case of a waveguide with only one sinusoidally undulating wall, by Samuels[2] for the case of a waveguide with inphase wall undulations, and by Salant[3] for the general problem. Nayfeh[4] showed that the above expansions are not uniform near the resonant frequencies because the correction term $\epsilon\phi_1$ dominates the first term ϕ_0. He determined a uniform expansion for waves propagating in a two-dimensional hard-walled duct with sinusoidally perturbed walls by using the method multiple scales.[5] In this paper, we extend the latter analysis to the case of linear waves propagating in a cylinderical hard-walled duct whose wall has weak undulations which need not be periodic.

The gas is assumed to be inviscid, irrotational, and nonheat conducting. Dimensionless quantities are introduced by using the mean radius of the duct r_0, the undisturbed speed of sound c, and the time r_0/c as reference quantities. The dimensionless radius of the duct is assumed to have the form

$$r(x)=1+\epsilon\sum_m \alpha_m \sin\kappa_m x \tag{1}$$

where ϵ is a small dimensionless constant parameter characterizing the weakness of the wall undulation and α_m and κ_m are constants; $r(x)$ need not be periodic.

The linearized equation describing the velocity potential $\tilde{\phi}(x,r,\theta,t)$ is

$$\nabla^2\tilde{\phi}-\tilde{\phi}_{tt}=0 \tag{2}$$

For harmonic time variations,

$$\tilde{\phi}(x,r,\theta,t)=\phi(x,r,\theta)\exp(-i\omega t) \tag{3}$$

where ω is the dimensionless frequency of oscillation. Substituting Eq. (3) into Eq. (2), we obtain the Helmholtz equation

$$\nabla^2+\omega^2\phi=0 \tag{4}$$

The flow-tangency conditon gives

$$\phi_r=\epsilon\phi_x\sum_m \alpha_m\kappa_m\cos\kappa_m x \quad \text{at} \quad r=1+\epsilon\sum_m \alpha_m\sin\kappa_m x \tag{5}$$

The straightforward expansion of the solution of Eqs. (4) and (5) is

$$\begin{aligned}\phi(x,r,\theta)=&\sum_{n,s} A_{ns}J_n(\beta_{ns}r)\exp[i(k_{ns}x+n\theta)]\\&+\frac{1}{2}i\epsilon\sum_{n,s,m}\alpha_m A_{ns}\{(k_{ns}\kappa_m-\beta_{ns}^2+n^2)\\&\times\Phi_{1_{mns}}(r)\exp[i(k_{ns}+\kappa_m)x+in\theta]\\&+(k_{ns}\kappa_m+\beta_{ns}^2-n^2)\Phi_{2_{mns}}(r)\exp[i(k_{ns}-\kappa_m)x+in\theta]\}\end{aligned} \tag{6}$$

where

$$\Phi_{1_{mns}}(r)=[J_n(\beta_{ns})/\gamma_{1_{mns}}J_n'(\gamma_{1_{mns}})]J_n(\gamma_{1_{mns}}r) \tag{7}$$

$$\Phi_{2_{mns}}(r)=[J_n(\beta_{ns})/\gamma_{2_{mns}}J_n'(\gamma_{2_{mns}})]J_n(\gamma_{2_{mns}}r) \tag{8}$$

$$k_{ns}^2=\omega^2-\beta_{ns}^2 \tag{9}$$

$$\gamma_{1_{mns}}^2=\omega^2-(k_{ns}+\kappa_m)^2 \tag{10}$$

$$\gamma_{2_{mns}}^2=\omega^2-(k_{ns}-\kappa_m)^2 \tag{11}$$

and the β_{ns} are the zeros of $J_n'(\beta)=0$. The solution given by Eqs. (6-11) is invalid if $\gamma_{1_{mns}}$ or $\gamma_{2_{mns}}$ approaches any β_{np} because the first-order term, correction term, in Eq. (6) dominates the zeroth-order term. This situation corresponds to the resonant frequency ω_r given by

$$\beta_{np}^2=\omega_r^2-(k_{ns}\pm\kappa_m)^2 \tag{12}$$

since $k_{np}^2=\omega_r^2-\beta_{np}^2$, it follows from Eq. (12) that

$$\kappa_m=k_{ns}\pm k_{np} \tag{13}$$

In other words, the straightforward expansion is not valid whenever a wavenumber of the wall undulations is near the sum or difference of the wavenumbers of any two duct modes. Nayfeh[4] showed that, if this condition is satisfied or nearly satisfied, these modes strongly interact and neither of them can propagate in the duct without strongly exciting the other. We note that, for the general wall undulation of Eq. (1), a given mode may be involved in more than one resonant condition.

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*University Distinguished Professor of Engineering. Member AIAA.

†Assistant Professor, Engineering Science and Mechanics Dept. Member AIAA.

In this paper, we describe the application of the method of multiple scales to this problem. The case of one resonant condition (two interacting modes) and the case of simultaneous resonant conditions (more than two interacting modes) are analyzed. This includes the conditions for cutoff waves.

II. Method of Solution

To determine a first-order uniform solution to Eqs. (4) and (5), we use the method of multiple scales and seek an expansion in the form

$$\phi(x,r,\theta;\epsilon)=\phi(x_0,x_1,r,\theta;\epsilon)=\phi_0(x_0,x_1,r,\theta)+\epsilon\phi_1(x_0,x_1,r,\theta) \tag{14}$$

where $x_0=x$ is a short scale characterizing the wavelengths of the propagating modes and $x_1=\epsilon x$ is a long scale characterizing the spatial modulations of the amplitudes and the phases of the modes. In terms of x_0 and x_1, the derivatives with respect to x become

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x_0}+\epsilon\frac{\partial}{\partial x_1},\quad \frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial x_0^2}+2\epsilon\frac{\partial^2}{\partial x_0\partial x_1}+\ldots \tag{15}$$

Substituting Eq. (14) into Eq. (4), using Eq. (15), and equating coefficients of like powers of ϵ, we obtain

$$\frac{\partial^2\phi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi_0}{\partial\theta^2}+\frac{\partial^2\phi_0}{\partial x_0^2}+\omega^2\phi_0=0 \tag{16}$$

$$\frac{\partial^2\phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi_1}{\partial\theta^2}+\frac{\partial^2\phi_1}{\partial x_0^2}+\omega^2\phi_1=-2\frac{\partial^2\phi_0}{\partial x_0\partial x_1} \tag{17}$$

Substituting Eq. (14) into Eq. (5), using Eq. (15), expanding the result in Taylor series about $r=1$ to transfer the boundary condition from

$$r=1+\epsilon\sum_m\alpha_m\sin\kappa_m x_0 \quad\text{to}\quad r=1$$

and equating coefficients of like powers of ϵ, we obtain

$$\frac{\partial\phi_0}{\partial r}=0 \quad\text{at}\quad r=1 \tag{18}$$

$$\frac{\partial\phi_1}{\partial r}=\frac{\partial\phi_0}{\partial x_0}\sum_m\alpha_m\kappa_m\cos\kappa_m x_0-\frac{\partial^2\phi_0}{\partial r^2}\sum_m\alpha_m\sin\kappa_m x_0 \quad\text{at}\quad r=1 \tag{19}$$

The solution of Eqs. (16) and (18) that is bounded at $r=0$ can be written as

$$\phi_0=\sum_{n,s}A_{ns}(x_1)J_n(\beta_{ns}r)\exp[1(k_{ns}x_0+n\theta)] \tag{20}$$

where

$$\omega^2-k_{ns}^2=\beta_{ns}^2 \tag{21}$$

the β_{ns} are the roots of

$$J_n'(\beta)=0 \tag{22}$$

and primes denote differentiation with respect to the argument. The A_{ns} are unknown functions at this level of approximation; they are determined by imposing the solvability condition at the next level of approximation.

Substituting for ϕ_0 from Eq. (20) into Eqs. (17) and (19) gives

$$\frac{\partial^2\phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi_1}{\partial\theta^2}+\frac{\partial^2\phi_1}{\partial x_0^2}+\omega^2\phi_1=-2i\sum_{n,s}A_{ns}'k_{ns}J_n(\beta_{ns}r)\exp[i(k_{ns}x_0+n\theta)] \tag{23}$$

$$\frac{\partial\phi_1}{\partial r}=\frac{1}{2}i\sum_{m,n,s}A_{ns}\alpha_m J_n(\beta_{ns}r)\,(k_{ns}\kappa_m-\beta_{ns}^2+n^2)\times\exp[i(k_{ns}+\kappa_m)x_0+in\theta]+(k_{ns}\kappa_m+\beta_{ns}^2-n^2)\exp[i(k_{ns}-\kappa_m)x_0+in\theta] \quad\text{at}\quad r=1 \tag{24}$$

Since the homogeneous parts of Eqs. (23) and (24) have the same form as Eqs. (16) and (18) and since the latter equations have a nontrivial solution, the inhomogeneous Eqs. (23) and (24) have a solution if, and only if, solvability conditions are satisfied. To determine these solvability conditions, we need to specify the resonant conditions that exist in the duct. In the following section, we consider the case of only two interacting modes, while in Section IV we consider three interacting modes.

III. Two Interacting Modes

In this section, we consider the interaction of the njth and nqth modes and assume that there are no other interactions. Thus, we let

$$k_{nq}-k_{nj}=\kappa_p+\epsilon\delta_1,\quad \delta_1=O(1) \tag{25}$$

Moreover, we write

$$(k_{nq}-\kappa_p)x_0=k_{nj}x_0+\epsilon\delta_1x_0=k_{nj}x_0+\delta_1x_1$$
$$(k_{nj}+\kappa_p)x_0=k_{nq}x_0-\epsilon\delta_1x_0=k_{nq}x_0-\delta_1x_1 \tag{26}$$

To determine the solvability conditions of Eqs. (23) and (24), we seek a particular solution of the form

$$\phi_1(x_0,x_1,r,\theta)=\sum_{n,s}\Phi_{ns}(x_1,r)\exp[i(k_{ns}x_0+n\theta)] \tag{27}$$

Substituting Eq. (27) into Eqs. (23) and (24) and equating the coefficients of like powers of $\exp[i(k_{ns}x_0+n\theta)]$ on both sides, we obtain

$$\frac{\partial^2\Phi_{ns}}{\partial r^2}+\frac{1}{2}\frac{\partial\xi\phi_{ns}}{\partial r}+\left(\beta_{ns}^2-\frac{n^2}{r^2}\right)\Phi_{ns}=-2ik_{ns}A_{ns}'(x_1)J_n(\beta_{ns}r) \tag{28}$$

$$\frac{\partial\Phi_{ns}}{\partial r}=0 \quad\text{at}\quad r=1 \quad\text{for}\quad s\neq j \text{ and } q \tag{29}$$

$$\frac{\partial\Phi_{nj}}{\partial r}=\frac{1}{2}i\alpha_pA_{nq}(k_{nq}\kappa_p+\beta_{nq}^2-n^2)J_n(\beta_{nq})\times\exp(i\delta_1x_1) \quad\text{at}\quad r=1 \tag{30}$$

$$\frac{\partial\Phi_{nq}}{\partial r}=\frac{1}{2}i\alpha_pA_{nj}(k_{nj}\kappa_p-\beta_{nj}^2+n^2)J_n(\beta_{nj})\times\exp(i\delta_1x_1) \quad\text{at}\quad r=1 \tag{31}$$

Thus, the problem of determining the solvability conditions of Eqs. (23) and (24) is transformed into determining the solvability conditions of Eqs. (28-31).

To determine the solvability condition of Eqs. (28-31), we multiply Eq. (28) by $rv(r)$, where $v(r)$ is specified later, integrate the result by parts from $r=0$ to $r=1$ to transfer the r-derivatives from Φ_{ns} to $v(r)$, and obtain,

$$\left[rv(r)\frac{\partial\Phi_{ns}}{\partial r}-r\Phi_{ns}v'(r)\right]_{r=0}^{r=1}+\int_0^1 r\Phi_{ns}\left[v''+\frac{1}{r}v'+\left(\beta_{ns}^2-\frac{n^2}{r^2}\right)v\right]\mathrm{d}r=-2ik_{ns}A'_{ns}\int_0^1 rJ_n(\beta_{ns}r)v(r)\mathrm{d}r \quad (32)$$

We choose $v(r)$ to be a solution of the so-called adjoint homogeneous problem

$$v''+\frac{1}{r}v'+\left(\beta_{ns}^2-\frac{n^2}{r^2}\right)v=0 \quad (33)$$

$$v'(1)=0, \quad v(0)<\infty \quad (34)$$

Hence, $v(r)=J_n(\beta_{ns}r)$. With this choice for $v(r)$, Eq. (32) becomes

$$J_n(\beta_{ns})\frac{\partial\Phi_{ns}}{\partial r}(x_1,1)=-ik_{ns}A'_{ns}\left(1-\frac{n^2}{\beta_{ns}^2}\right)J_n(\beta_{ns}) \quad (35)$$

Combining Eqs. (29) and (35), we find that

$$A'_{ns}=0 \text{ or } A_{ns}=\text{const. for } s\neq j \text{ and } q \quad (36)$$

Therefore, these modes propagate unaffected along the duct. Combining Eq. (35) with $s=j$ and Eq. (30), we have

$$A'_{nj}=-\frac{\alpha_p J_n(\beta_{nq})(k_{nq}\kappa_p+\beta_{nq}^2-n^2)}{2k_{nj}J_n(\beta_{nj})(1-n^2/\beta_{nj}^2)}A_{nq}\exp(i\delta_1 x_1) \quad (37)$$

Similarly, it follows from Eq. (35) with $s=q$ and Eq. (31) that

$$A'_{nq}=-\frac{\alpha_p J_n(\beta_{nj})(k_{nj}\kappa_p-\beta_{nj}^2+n^2)}{2k_{nq}J_n(\beta_{nq})(1-n^2/\beta_{nq}^2)}A_{nj}\exp(-i\delta_1 x_1) \quad (38)$$

Equations (37) and (38) have solutions of the form

$$A_{nj}=a_j\exp(i\gamma x_1) \text{ and } A_{nq}=a_q\exp[i(\gamma-\delta_1)x_1] \quad (39)$$

where a_j and a_q are constants. Substituting Eq. (39) into Eqs. (37) and (38) gives

$$\gamma^2-\delta_1\gamma+\Omega=0 \quad (40)$$

where

$$\Omega=\frac{\alpha_p^2(k_{nq}\kappa_p+\beta_{nq}^2-n^2)(k_{nj}\kappa_p-\beta_{nj}^2+n^2)}{4k_{nj}k_{nq}(1-n^2/\beta_{nj}^2)(1-n^2/\beta_{nq}^2)} \quad (41)$$

The solution of Eq. (40) is given by

$$\gamma=(\delta_1/2)\pm\sqrt{(\delta_1/2)^2-\Omega} \quad (42)$$

If the roots of Eq. (42) are real, then the two modes propagate unattenuated. It can be shown that this is the case if the two modes are simultaneously propagating downstream or upstream. For the case of axisymmetric modes ($n=0$), Eq. (41) reads

$$\Omega=(k_{0_q}\kappa_p+\beta_{0_q}^2)(k_{0_j}\kappa_p-\beta_{0_j}^2)/4k_{0_j}k_{0_q} \quad (43)$$

which is negative because $k_{0_j}\kappa_p-\beta_{0_j}^2$ is negative as shown below

$$\begin{aligned}k_{0_j}\kappa_p-\beta_{0_j}^2&=k_{0_j}\kappa_p-\omega^2+k_{0_j}^2\\&=k_{0_j}\kappa_p-k_{0_q}^2-\beta_{0_q}^2+k_{0_j}^2\\&=k_{0_j}\kappa_p-(\kappa_p-\epsilon\delta_1)(k_{0_q}+k_{0_j})-\beta_{0_q}^2\\&=-(k_{0_q}\kappa_p+\beta_{0_q}^2)+\epsilon\delta02(k_{0_q}+k_{0_j})<0\end{aligned} \quad (44)$$

On the other hand, if the njth mode is propagating downstream and the nqth mode is propagating upstream, then for $n=0$, Eq. (43) shows that Ω is positive and the roots given by Eq. (42) are complex. Therefore, the modes propagate attenuated when $\Omega>\frac{1}{4}\delta^2$ and hence they are cut off.

To find the κ_p range within which two oppositely propagating modes are cut off, we equate the discriminant of Eq. (42) to zero and obtain

$$(\delta_1/2)^2=\Omega \quad (45)$$

Substituting for δ_1 from Eq. (25) into Eq. (45) and using Eq. (41), we obtain the following equation for κ_p:

$$A\kappa_p^2+2B\kappa_p+c=0 \quad (46)$$

where

$$A=1-\Delta \quad (47)$$

$$B=k_{nj}-k_{nq}+(\Delta/2k_{nj}k_{nq}(\beta_{nj}^2-n^2)(\beta_{nq}^2-n^2)] -k_{nj}(\beta_{nq}^2-n^2)] \quad (48)$$

$$R=(k_{nj}-k_{nq})^2+(\Delta/k_{nj}k_{nq})(\beta_{nj}^2-n^2)(\beta_{nq}^2-n^2) \quad (49)$$

$$\Delta=\alpha_p^2\epsilon^2/[(1-n^2/\beta_{nj}^2)(1-n^2/\beta_{nq}^2)] \quad (50)$$

For $\omega=10$, $\alpha=1$, $\epsilon=0.05$, Table 1 shows the κ_p range within which two oppositely propagating modes are cut off.

This shows that the njth mode cannot exist without strongly generating the nqth mode. Both modes travel unattenuated through the duct if they are propagating in one direction. However, if the two interacting modes are propagating in opposite directions, then they may be cut off depending on the detuning.

Table 1 κ_p **range for which two oppositely propagating modes are cut off**

n	$\beta_{n1}\beta_{n2}$	k_n, k_n	κ_p range
0	3.83,7.016	−9.237,7.126	15.418-17.412
1	5.136,8.417	−8.581,5.399	13.025-15.148

IV. Three Interacting Modes

In this section, we consider the interaction of the njth, nqth, and ntth modes and assume that there are not other interactions. Thus, we let

$$k_{nq}-k_{nj}=\kappa_p+\epsilon\delta_1, \quad \delta_1=O(1) \quad (51)$$

and

$$k_{nt}-k_{nq}=\kappa_u+\epsilon\delta_2, \quad \delta_2=O(1) \quad (52)$$

We write

$$(k_{nq}-\kappa_p)x_0=k_{nj}x_0+\epsilon\delta_1 x_0=k_{nj}x_0+\delta_1 x_1 \quad (53)$$

$$(k_{nj}+\kappa_p)x_0=k_{nq}x_0-\epsilon\delta_1 x_0=k_{nq}x_0-\delta_1 x_1 \quad (54)$$

$$(k_{nt}-\kappa_u)x_0=k_{nq}x_0+\epsilon\delta_2 x_0=k_{nq}x_0+\delta_2 x_1 \quad (55)$$

$$(k_{nq}+\kappa_u)x_0=k_{nt}x_0-\epsilon\delta_2 x_0=k_{nt}x_0-\delta_2 x_1 \quad (56)$$

To determine the solvability conditions of Eqs. (23) and (24), we substitute Eq. (27) into these equations and equate coefficients of like powers of $\exp[i(k_{ns}x_0+n\theta)]$ on both sides, and obtain

$$\frac{\partial^2\Phi_{ns}}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi_{ns}}{\partial r}+\left(\beta_{ns}^2-\frac{n^2}{r^2}\right)\Phi_{ns}=-2iA'_{ns}(x_1)k_{ns}J_n(\beta_{ns}r) \tag{57}$$

$$\frac{\partial\Phi_{ns}}{\partial r}=0 \text{ at } r=1 \text{ for } s\neq j,\ q \text{ and } t \tag{58}$$

$$\frac{\partial\Phi_{nj}}{\partial r}=\frac{1}{2}i\alpha_pA_{nq}(k_{nq}\kappa_p+\beta_{nq}^2-n^2)J_n(\beta_{nq})\times\exp(i\delta_1x_1) \text{ at } r=1 \tag{59}$$

$$\frac{\partial\Phi_{nq}}{\partial r}=\frac{1}{2}i\alpha_pA_{nj}(k_{nj}\kappa_p-\beta_{nj}^2+n^2)J_n(\beta_{nj})\times\exp(-i\delta_1x_1)+\frac{1}{2}i\alpha_uA_{nt}(k_{nt}\kappa_u+\beta_{nt}^2-n^2)J_n(\beta_{nt})\exp(i\delta_2x_1) \text{ at } r=1 \tag{60}$$

$$\frac{\partial\Phi_{nt}}{\partial r}=\frac{1}{2}i\alpha_uA_{nq}(k_{nq}\kappa_u-\beta_{nq}^2+n^2)J_n(\beta_{nq})\times\exp(-i\delta_2x_1) \text{ at } r=1 \tag{61}$$

The solvability condition of Eq. (57) yields Eq. (35). Combining Eq. (35) with $s=j$ and Eq. (59) we have

$$A'_{nj}=-\frac{\alpha_pJ_n(\beta_{nq})(k_{nq}\kappa_p+\beta_{nq}^2-n^2)}{2k_{nj}J_n(\beta_{nj})(1-n^2/\beta_{nj}^2)}A_{nq}\exp(i\delta_1x_1) \tag{62}$$

Similarly, it follows from Eq. (35) with $s=q$ and Eq. (60) that

$$A'_{nq}=-\frac{\alpha_pJ_n(\beta_{nj})(k_{nj}\kappa_p-\beta_{nj}^2+n^2)}{2k_{nq}J_n(\beta_{nq})(1-n^2/\beta_{nq}^2)}A_{nj}\exp(-i\delta_1x_1)-\frac{\alpha_uJ_n(\beta_{nt})(k_{nt}\kappa_u+\beta_{nt}^2-n^2)}{2k_{nq}J_n(\beta_{nq})(1-n^2/\beta_{nq}^2)}A_{nt}\exp(i\delta_2x_1) \tag{63}$$

Finally, it follows from Eq. (35) with $s=t$ and Eq. (61) that

$$A'_{nt}=-\frac{\alpha_uJ_n(\beta_{nq})(k_{nq}\kappa_u-\beta_{nq}^2+n^2)}{2k_{nt}J(\beta_{nt})(1-n^2/\beta_{nt}^2)}A_{nq}\exp(-i\delta_2x_1) \tag{64}$$

We note that Eq. (63) couples the three interacting modes. Equations (62-64) have solutions of the form

$$A_{nj}=a_j\exp(i\gamma x_1),\quad A_{nq}=a_q\exp[i(\gamma-\delta_1)x_1],\quad A_{nt}=a_t\exp[i(\gamma-\delta_1-\delta_2)x_1] \tag{65}$$

where a_j, a_q, and a_t are constants. Substituting Eq. (65) into Eqs. (62-64) gives

$$\gamma^3+\Omega_1\gamma^2+\Omega_2\gamma+\Omega_3=0 \tag{66}$$

where

$$\Omega_1=-(2\delta_1+\delta_2) \tag{67}$$

$$\Omega_2=\delta_1(\delta_1+\delta_2)+[1/4k_{nq}(1-n^2/\beta_{nq}^2)]\times[\alpha_u^2(k_{nt}\kappa_u+\beta_{nt}^2-n^2)(k_{nq}\kappa_u-\beta_{nq}^2+n^2)/k_{nt}(1-n^2/\beta_{nt}^2)+\alpha_p^2(k_{nq}\kappa_p+\beta_{nq}^2-n^2)\times(k_{nj}\kappa_p-\beta_{nj}^2+n^2)/k_{nj}(1-n^2/\beta_{nj}^2)] \tag{68}$$

$$\Omega_3=-(\delta_1+\delta_2)\alpha_p^2(k_{nq}\kappa_p+\beta_{nq}^2-n^2)(k_{nj}\kappa_p-\beta_{nj}^2+n^2)/[4k_{nj}k_{nq}(1-n^2/\beta_{nj}^2)(1-n^2/\beta_{nq}^2)] \tag{69}$$

If all the roots of Eq. (66) are real, none of the modes can exist without strongly exciting the others and the three modes propagate unattenuated (i.e., there is a passbond) while exchanging their energy in accordance with energy conservation. On the other hand, if two of the roots are complex, none of the modes can exist without strongly exciting the others and the three modes are cutoff (i.e., there is a stopband).

To determine the possibility of the existence of stopbands or cutoff, we let $\delta_1=\delta_2=0$ in Eqs. (66-69) and obtain

$$\gamma^3+\Omega_2\gamma=0 \tag{70}$$

where

$$\Omega_2=\alpha_u^2\Omega_{qt}+\alpha_p^2\Omega_{qj} \tag{71}$$

Here, Ω_{qj} is defined by Eq. (41) and Ω_{qt} is also defined by Eq. (41) if the subscript j is replaced by t and the subscript p is replaced by u. The roots of Eq. (70) are

$$\gamma=0,\quad \gamma=\pm(-\Omega_2)^{1/2} \tag{72}$$

Thus, stopbands are possible only if $\Omega_2>0$; otherwise, there is a passband. Inspection of Eq. (41) and the discussion in the preceding section show that $\Omega_{qj}<0$ if $k_{nq}k_{nj}>0$ (i.e., the qth and jth mode propagate in the same direction) and $\Omega_{qj}>0$ if $k_{nq}k_{nj}<0$ (i.e., the two modes propagate in opposite directions). Thus, there are three cases. First, the three interacting modes propagate in the same direction; then, Ω_{qj} and Ω_{qt} are negative, and the γ's are real, and the three modes propagate unattenuated (i.e., there is a passband). Second, the qth mode propagates opposite to the njth and ntth modes; then, Ω_{qj} and Ω_{qt} are positive, two of the γ's are complex, and the three modes are cutoff (i.e., there is a stopband). Third, either the njth or the ntth mode propagates opposite to the remaining modes; then $\Omega_{qj}\Omega_{qt}<0$, the γ's may be complex depending on the relative magnitudes of the Ω_{qj} and Ω_{qt} and the ratio of α_u to α_p. We note that in this case, one can always find a ratio α_u/α_p which makes $\Omega_2>0$, and hence cutoff the three modes.

When δ_1 and δ_2 are different from zero, one can generalize the preceding conclusions. When the three interacting waves propagate in the same direction, there is a passband and the three modes propagate unattenuated with the energy being continuously exchanged among them. When the nqth mode propagates opposite to the other two modes, there exists a frequency stopband. When either the njth or the ntth mode propagates opposite to the other two modes, one can always produce a frequency stopband by a proper choice of the amplitudes of the wall harmonics.

V. Concluding Remarks

An analysis is presented for the propagation of sound waves in ducts whose walls have general multiwavelength undulations. A harmonic wall undulation with the wavenumber κ_p couples two propagating modes with the wavenumbers k_q and k_j if $\kappa_p\approx k_q\pm k_j$. Results are presented when the wall undulations couple either two or three modes.

In the case of two interaction modes, there is a passband if they propagate in the same direction and there is a stopband if they propagate in opposite directions. In the case of three interacting modes, there is a passband if they propagate in the same direction, and one can always find a stopband if they do not propagate in the same direction by adjusting the relative magnitudes of the compoents of the wall undulations.

The analysis can be easily extended to the case of more than three interacting modes. For example, if there are four simultaneously interacting modes connected by

$$\kappa_1 = k_{nj} - k_{ni} + \epsilon\delta_1$$

$$\kappa_2 = k_{n\ell} - k_{nj} + \epsilon\delta_2$$

$$\kappa_3 = k_{nm} - k_{n\ell} + \epsilon\delta_3 \tag{73}$$

the equations describing the interaction have the form

$$A'_{ni} = -\Gamma_{ij} A_{nj} \exp(i\delta_1 x_1) \tag{74}$$

$$A'_{nj} = -\Gamma_{ji} A_{ni} \exp(-i\delta_1 x_1) - \Gamma_{j\ell} A_{n\ell} \exp(i\delta_2 x_1) \tag{75}$$

$$A'_{n\ell} = -\Gamma_{\ell j} A_{nj} \exp(-i\delta_2 x_1) - \Gamma_{\ell m} A_{nm} \exp(i\delta_3 x_1) \tag{76}$$

$$A'_{nm} = -\Gamma_{m\ell} A_{n\ell} \exp(-i\delta_3 x_1) \tag{77}$$

as in Eqs. (37) and (38).

The analysis can also be easily extended to the case of a wave packet centered about the frequency ω. In this case, an asymptotic expansion is sought for the potential function of Eq. (2) in the form

$$\tilde{\phi}(x,r,\theta,t;\epsilon) = \tilde{\phi}(x_0,x_1,r,\theta,t_0,t_1;\epsilon) = \tilde{\phi}_0(x_0,x_1,r,\theta,t_0,t_1) + \epsilon\tilde{\phi}(x_0,x_1,r,\theta,t_0,t_1) + \ldots \tag{78}$$

where $t_0 = t$ and $t_1 = \epsilon t$. The amplitude functions A_{ns} are now functions of x_1 and t_1. Applying the steps of the method of multiple scales, one obtains the following partial-differential equations for the case of three interacting modes:

$$\frac{\partial A_{nj}}{\partial x_1} + k'_{nj}\frac{\partial A_{nj}}{\partial t_1} = -\Gamma_{jq} A_{nq} \exp(i\delta_1 x_1) \tag{79}$$

$$\frac{\partial A_{nq}}{\partial x_1} + k'_{nq}\frac{\partial A_{nq}}{\partial t_1} = -\Gamma_{qj} A_{nk} \exp(-i\delta_1 x_1) - \Gamma_{qt} A_{nt} \exp(i\delta_2 x_1) \tag{80}$$

$$\frac{\partial A_{nt}}{\partial x_1} + k'_{nt}\frac{\partial A_{nt}}{\partial t_1} = -\Gamma_{tq} A_{nq} \exp(-i\delta_2 x_1) \tag{81}$$

where $k'_{ns} = \mathrm{d}k_{ns}/\mathrm{d}\omega$ is the inverse of the group velocity. The solution of these equations determines the spatial and temporal modulation of the propagating wave packet. In the case of two interacting modes (say njth and nqth), one can use the variables $\xi = t_1 - k'_{nj}x_1$ and $\eta = t_1 - k'_{nq}x_1$ and show that A_{nj} and A_{nq} are governed by a telegraph equation.

The present results can be used to reduce the noise in air-conditioning and forced-air heating units by deliberately building undulations in the duct walls. With the right undulations, one can cut off most, if not all, modes in such ducts. Moreover, the present analysis and results are also applicable to uniform ducts with liners whose admittances have small periodic components. With such liners, one can couple low-order modes which are least attenuated with high-order modes which are highly attenuated, thereby increasing the effectiveness of the liner. In addition, one can transfer part of the energy to backward propagating modes, thereby decreasing the radiated noise. Since almost all liners have some degree of nonuniformity, coupling of the different modes in the duct might exist. If the modes used to determine an optimum admittance are coupled with other modes, the full effectiveness of the liner may not be realized.

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